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Hidden and contact symmetries of ordinary differential equations

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Abstract. The Lie algebra for the maximal contact symmetries of third-order ordinary differential equations (ODEs) is examined for type I and II hidden symmetries where the analysis of hidden symmetries for point symmetries is extended to contact symmetries. ODEs invariant under the group associated with the ten-dimensional (maximal) Lie algebra may produce type I hidden symmetries for two-parameter subgroups and type II hidden symmetries for certain solvable non-Abelian three-parameter subgroups in the third-order ODEs when they are reduced in order. A new class of type II hidden symmetries is recognized in which contact symmetries transform to point symmetries in some reduction paths. Two examples of ODEs invariant under subgroups of the ten-parameter group under which $y''' = 0$ is invariant demonstrate the new class of type II hidden symmetries.

1. Introduction

Hidden symmetries of ordinary differential equations have recently been introduced [1–4] and analysed for various physical and mathematical equations [5–16]. These are symmetries not found by the Lie classical method for point symmetries of differential equations [1, 17–20]. Type I (II) hidden symmetries of ordinary differential equations (ODEs) have been defined as point symmetries that are lost (are gained) when the order of the ODE is reduced by the invariants of a Lie group symmetry. Hidden symmetries of partial differential equations (PDEs) also exist but are not discussed here [10, 21]. The invariants of symmetries of ODEs have been found from the characteristic equations of the local (point) group generators but invariants have also been found from exponential non-local group generators [6, 9, 11, 16]. For ODEs these invariants are usually called differential invariants but the two lowest order invariants are the path curve, a function of the independent and dependent variable, and the first differential invariant, a function of the independent variable, the dependent variable and the derivative of the dependent variable with respect to the independent variable.

The investigations of hidden symmetries have focused on the reduction of order of ODEs by additional and longer reduction paths and on understanding the origin of these

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hidden symmetries. A motivation for studying these hidden symmetries arose from the nonlinear characteristic equations of the Vlasov equation for plasmas for which additional solutions would be advantageous [22]. The existence of type I hidden symmetries allows the increase in the order of an ODE to a new ODE of one higher order which has more than one additional point symmetry. This *ad hoc* approach requires guessing the variable transformation. In a more systematic approach an ODE can be reduced in order by non-normal subgroup invariants such that one or more point symmetries of the ODE are lost in addition to the symmetry used in the reduction of the ODE. The lower-order reduced ODE has a type I hidden symmetry and tables can be prepared of these ODEs [2,4]. From the tables the order of the ODEs in the table can be increased to the higher-order ODE with the additional symmetries. For both the *ad hoc* and systematic methods an ODE can be reduced in order more than would be expected by the point symmetries determined by the Lie classical method. Type II hidden symmetries also extend the reduction path to an ODE of lower-order than expected by the symmetries determined of the original ODE by the Lie classical method since one or more symmetries are gained when the order of this ODE is reduced.

The origin of type I hidden symmetries is known. These arise if the order of an ODE is reduced by the invariants of a non-normal subgroup. The existence of a non-normal subgroup can be identified from the commutation relations of the Lie algebra of the group generators which represent the symmetry under which the ODE is invariant [1, 19]. A normal subgroup has the infinitesimal (group) generator U_i which satisfies the commutator

$$[U_i, U_j] = C_{ij}^k U_k \tag{1}$$

for $j \neq i$ where either $k = i$ or $C_{ij}^k = 0$. If $j \neq i$, $k \neq i$ and $C_{ij}^k \neq 0$, the point symmetries of U_j vanish when the order of the ODE is reduced by the non-normal subgroup invariants (path curve and first differential invariant) of U_i.

The origin of type II hidden symmetries has until recently been obscure. One class of type II hidden symmetries arises when a third-order ODE, invariant under a solvable, non-Abelian three-parameter Lie group with at least one commutator that involves the three group generators, is reduced in order by the appropriate non-normal subgroup invariants [9, 14]. The two distinct Lie algebras are:

Case 1

$$[U_i, U_j] = C_{ij}^k U_k \qquad [U_j, U_k] = C_{jk}^i U_i \qquad [U_k, U_i] = 0 \tag{2}$$

Case 2

$$[U_i, U_j] = C_{ij}^k U_k \qquad [U_j, U_k] = C_{jk}^k U_k \qquad [U_k, U_i] = 0 \tag{3}$$

where the repeated indices do not indicate summation. The generator basis of case 2 may be modified such that two commutators are zero. It is assumed that the order of the third-order ODE is reduced by the invariants of U_i and that the order of the second-order ODE is reduced by the invariants of G_k (the reduced form of U_k) for these two cases. Then C_{ij}^k is always non-zero but C_{jk}^i or C_{jk}^k may be zero. The reduction of the third-order ODE transforms U_j to a linear non-local group generator and U_k to a local group generator. This has been shown by assuming the normal forms for U_i and U_k [14] and by solving for the general form of U_j. The reduction of the second-order ODE by the invariants of the reduced form of U_k transforms the reduced form of U_j to a local group generator. For case 1 we could just as well reduce the order of the third-order ODE by U_k. For case 2 reduction by U_k first gives only local group generators in the subsequent reductions. Reduction by U_j first gives non-local group generators in both cases for U_i and U_k if both structure constants are non-zero. We note that, if both structure constants are non-zero in case I, that there is no path that has local group generators in all reductions, but the group is nevertheless solvable.

Solvable structures have been applied to type II hidden symmetries [8, 12] that appeared in two examples [1, 3] of second-order ODEs. The solvable structures approach in these references substitutes a linear operator for the second-order ODE as has been done in an alternate approach to symmetry analysis of ODEs [19, 23]. In addition to the point group generator, that represents the symmetries of this ODE, it introduces a group generator of the reduced ODE which reveals the existence of the hidden symmetry. The PDE for the coordinate function of this latter group generator is the determining equation found by applying the Lie classical method to the first-order ODE. The difficulties usually associated with solving this determining equation for the invariance of a first-order ODE are avoided by the choice of a single coordinate function. More general forms of the group generator of the first-order ODE that represent the type II hidden symmetry are possible but can be difficult to calculate. Type II hidden symmetries are not analysed by the method of solvable structures in this paper.

Third-order ODEs and contact symmetries are discussed in section 2. The commutation table of the Lie algebra for a $y''' = 0$ found previously [15] is given in section 3 and the reduction of ODEs by use of these tables is discussed. A new class of type II hidden symmetries is identified in section 3. In section 4 a second-order ODE with the new type II hidden symmetry is analysed. In section 5 a third-order ODE which has two classes of type II hidden symmetries is analysed. Section 6 presents the conclusions.

2. Third-order ODEs and contact symmetries

Contact transformations are transformations in which the transformed variables $\bar{x}$, $\bar{y}$, and $\bar{y}'$ are functions of the original variables x, y and y' [19]. The infinitesimal transformation is represented by the group generator which has the form

$$U_j = \xi_j(x, y, y')\frac{\partial}{\partial x} + \eta_j(x, y, y')\frac{\partial}{\partial y} + \eta_j'(x, y, y')\frac{\partial}{\partial y'} \tag{4}$$

where

$$\eta_j' = \frac{d\eta_j}{dx} - \frac{d\xi_j}{dx}y' \qquad \text{and} \qquad \frac{\partial \eta_j}{\partial y'} = \frac{\partial \xi_j}{\partial y'}y'. \tag{5}$$

These relations are explained by Stephani [19] and in other references. Customarily the two conditions in equation (5) are then replaced by the coordinate functions defined in terms of a characteristic (generating) function $\Omega_j(x, y, y')$ which are

$$\xi_j = \frac{\partial \Omega_j}{\partial y'} \qquad \eta_j = y'\frac{\partial \Omega_j}{\partial y'} - \Omega_j \qquad \eta_j' = -\frac{\partial \Omega_j}{\partial x} - y'\frac{\partial \Omega_j}{\partial y}. \tag{6}$$

If the coordinate functions ξ_j and η_j depend on x and y only, the contact symmetries are identical to point symmetries. To differentiate between contact symmetries that are identical with point symmetries and those that are not we define a contact symmetry where at least one of ξ_j and η_j depends on y' as an intrinsic contact symmetry.

Computing the contact symmetries for an ODE by a direct method is done in principle in the same fashion as for the Lie classical method for point symmetries. The contact symmetries for first-order ODEs are essentially the same as point symmetries but for second-order ODEs there are, in principle, an infinite number of contact symmetries where the determining equation is a single PDE in x, y and y'. This single PDE is even more complicated to solve than the single determining equation for point symmetries of a first-order ODE. The determining equation for a third-order ODE does separate into a set of differential equations. The method employed here for finding the contact symmetries of a third-order ODE avoids the introduction of the generating function $\Omega_j(x, y, y')$ and uses the symbolic program LIE [24]. The third-order ODE is replaced by three lower-order ODEs where y' is replaced by a third variable, v. The point symmetries which are determined by LIE for this set of ODEs are the contact symmetries. All contact and generalized symmetries may be found in principle if the third-order ODE is replaced by three first-order ODEs [1] but finding the symmetries of a set of first-order ODEs is not simple. Usually an ansatz of the dependence of the coordinate functions on the variables must be made [25]. For contact symmetries of a third-order ODE

$$y''' = f(x, y, y', y'') \tag{7}$$

we let $v = y'$ such that

$$v'' = f(x, y, v, v') \tag{8a}$$

$$v = y' \tag{8b}$$

$$v' = y'' \tag{8c}$$

where the prime denotes differentiation with respect to x and equation (8c) is necessary so that the computer program does not treat v' and y'' as independent variables. The coordinate functions as found by the symbolic program LIE for the point symmetries of equations (8a)–(8c) are now functions of (x, y, v). These coordinate functions can be expressed as functions of (x, y, y') for the contact symmetries of the original third-order ODE (7).

3. Lie algebra of contact symmetries for third-order ODEs

The commutation tables for two third-order ODEs have been reported [15]. The commutation table for the ODE

$$y''' = 0 \tag{9}$$

is given in table 1. The group generators for the ten contact symmetries in different form given by Ibragimov [20] are

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial y} & G_6 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial y'} \\
G_2 &= x\frac{\partial}{\partial y} + \frac{\partial}{\partial y'} & G_7 &= y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} \\
G_3 &= x^2\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial y'} & G_8 &= y'\frac{\partial}{\partial x} + \frac{1}{2}y'^2\frac{\partial}{\partial y} \\
G_4 &= \frac{\partial}{\partial x} & G_9 &= 2(xy' - y)\frac{\partial}{\partial x} + xy'^2\frac{\partial}{\partial y} + y'^2\frac{\partial}{\partial y'} \\
G_5 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} & G_{10} &= (x^2y' - 2xy)\frac{\partial}{\partial x} + \left(\frac{1}{2}x^2y'^2 - 2y^2\right)\frac{\partial}{\partial y} \\
& & &\quad + (xy'^2 - 2yy')\frac{\partial}{\partial y'}.
\end{aligned} \tag{10}$$

The contact symmetries of this Lie algebra $sp(4)$ [15] can be further divided into seven point symmetries, G_1–G_7, and three intrinsic contact symmetries, G_8–G_{10}, where all point symmetries are also contact symmetries. The symmetries, G_8–G_{10}, can be verified to be intrinsic contact symmetries by finding the characteristic function from equation (6).

Table 1. Commutation table for the ten contact symmetries of $y''' = 0$.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}
G_1	0	0	0	0	G_1	$2G_2$	G_1	0	$-2G_4$	$-2(G_5+G_7)$
G_2	0	0	0	$-G_1$	0	G_3	G_2	G_4	$2G_7$	$-G_6$
G_3	0	0	0	$-2G_2$	$-G_3$	0	G_3	$2(G_5-G_7)$	$2G_6$	0
G_4	0	G_1	$2G_2$	0	G_4	$2G_5$	0	0	$2G_8$	G_9
G_5	$-G_1$	0	G_3	$-G_4$	0	G_6	0	$-G_8$	0	G_{10}
G_6	$-2G_2$	$-G_3$	0	$-2G_5$	$-G_6$	0	0	$-G_9$	$-2G_{10}$	0
G_7	$-G_1$	$-G_2$	$-G_3$	0	0	0	0	G_8	G_9	G_{10}
G_8	0	$-G_4$	$-2(G_5-G_7)$	0	G_8	G_9	$-G_8$	0	0	0
G_9	$2G_4$	$-2G_7$	$-2G_6$	$-2G_8$	0	$2G_{10}$	$-G_9$	0	0	0
G_{10}	$2(G_5+G_7)$	G_6	0	$-G_9$	$-G_{10}$	0	$-G_{10}$	0	0	0

The Kummer–Schwarz equation

$$2y'y''' - 3y''^2 = 0 \tag{11}$$

is also invariant under the ten-dimensional Lie algebra $sp(4)$ for the contact symmetries. The ten contact symmetries for the Kummer–Schwarz equation further divide into six point symmetries, and four intrinsic contact symmetries. The commutation table for the Kummer–Schwarz equation has the same form as that in table 1 but the group generators in table 1 are replaced by other group generators that are given elsewhere [15].

The conditions for type I and type II hidden symmetries as given for the Lie point symmetries apply also to contact symmetries. That can be seen since the contact symmetries of the third-order ODE (7) are point symmetries of the equivalent set of one second-order and two first-order ODEs in equations (8*a*)–(8*c*).

The reduction of the equation (9) to quadratures can be done by using the symmetries of the Abelian subalgebras of G_1, G_2 and G_3 or G_8, G_9 and G_{10} in any order. The symmetries of the group generators G_4, G_5 and G_6 cannot be used to reduce equation (9) to quadratures since the subgroup represented by these group generators is not solvable. Other reduction paths may take various symmetries. Let us examine one case. Reduce equation (9), $y''' = 0$, by the invariants, $u = x$, $v = y'$, of G_1 to find

$$\ddot{v} = 0 \tag{12}$$

where the overdot denotes differentiation with respect to u. The conditions for the group generators of the reduced ODE to become non-local are given in equation (1). In this reduction the symmetries of G_6, G_9 and G_{10} become non-local. The reduced group generators of G_2, G_3, G_4, G_5, G_7 and G_8 are local. In addition we note that the contact group generator G_8 transforms to a point group generator when reduced. The reduced group generators can be found from the relation

$$V_j = G_j(u)\frac{\partial}{\partial u} + G_j(v)\frac{\partial}{\partial v} \tag{13}$$

for $j = 2$ to 10 where we already have the once extended group generators of the point symmetries since the group generators in equation (10) represent contact symmetries. The reduced group generators in the invariants, u and v, are

$$V_2 = \frac{\partial}{\partial v} \qquad\qquad V_7 = v\frac{\partial}{\partial v}$$

$$V_3 = 2u\frac{\partial}{\partial v} \qquad V_8 = v\frac{\partial}{\partial u}$$

$$V_4 = \frac{\partial}{\partial u} \qquad V_9 = -2\left(uv - \int v\,\mathrm{d}u\right)\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v}$$

$$V_5 = u\frac{\partial}{\partial u} \qquad V_{10} = \left(u^2 v - 2u\int v\,\mathrm{d}u\right)\frac{\partial}{\partial u} + \left(uv^2 - 2v\int v\,\mathrm{d}u\right)\frac{\partial}{\partial v}.$$

$$V_6 = u^2\frac{\partial}{\partial x} + 2\int v\,\mathrm{d}u\frac{\partial}{\partial v} \tag{14}$$

The third-order ODE (9) has the maximum number of point symmetries which is seven. One of those point symmetries is used to reduce the order of the ODE, one point symmetry becomes non-local and one contact symmetry becomes a point symmetry which leaves six point symmetries of the second-order ODE (12). However Lie [19, 23] proved that the second-order ODE, $\ddot{v} = 0$, has eight point symmetries. The other two point symmetries are type II hidden symmetries, i.e. their origins lie in non-local symmetries of (9). (In the case of a non-Cartan symmetry its origin lies in a non-local contact symmetry.)

The transformation of G_8 from an intrinsic contact group generator to a point group generator upon the reduction of order of the third-order ODE (9) introduces a new class of type II hidden symmetries. The other two contact symmetries as represented by G_9 and G_{10} transform to non-local symmetries. The intrinsic contact symmetries all reduce to symmetries that are something other than intrinsic contact symmetries. This is expected since as y' is the new dependent variable, the coordinate functions of the reduced group generators depend on the new variables u and v and possibly the non-local variable x. Many examples of intrinsic contact symmetries transforming to point symmetries upon the reduction of order of the ODE by symmetry invariants can be found by using the symmetries of the group $Sp(4)$. A complete investigation of all the reduction paths that are possible of equation (9) with the symmetries of $Sp(4)$ is clearly formidable. Nevertheless, if that were done, a number of lower-order ODEs with varying number of symmetries could be found. The reduction of the Kummer–Schwarz equation proceeds in the same manner. The significant result in comparing the symmetries of equations (9) and (11) is that the point symmetries differ but the Lie algebra of the ten contact symmetries is the same. As a consequence the reduction in order by symmetries of these two equations should be similar.

4. Reduction of ODE invariant under one point and one contact symmetry

For the first example we consider a second-order ODE that is invariant under one Lie point symmetry and one contact symmetry. The group generators are G_4 (point) and G_8 (contact) from equation (10) and they commute. The general form of a second-order ODE invariant under the two groups represented by these group generators is

$$y'' = \frac{y'^2}{2y + y'^2 g(y')} \tag{15}$$

where $g(y')$ is an arbitrary function of y'. The ODE (15) has a type II hidden symmetry since the Lie classical method finds only one Lie point symmetry. The other symmetry can be verified as an intrinsic contact symmetry by finding the generating function from equation (6) for G_8. The reduction is done by the invariants of the point symmetries of G_4 first although we could do the reduction by G_8 first since the two-parameter subgroup is

Abelian. The invariants are $u = y$, $v = y'$. The reduced first-order ODE is

$$v\dot{v} = \frac{v^2}{2u + v^2 g(v)} \tag{16}$$

where $\dot{v} = \mathrm{d}v/\mathrm{d}u$. The reduced group generator V_8 is found to be

$$V_8 = G_8^{(1)}(u)\frac{\partial}{\partial u} + G_8^{(1)}(v)\frac{\partial}{\partial v} = \frac{1}{2}v^2\frac{\partial}{\partial u} \tag{17}$$

where the contact group generator G_8 has been transformed upon reduction to a local (point) group generator V_8. That occurs since the differential invariants of G_8 define new coordinates $u(x, y)$ and $v(x, y, y')$ such that the derivative y' is folded into the dependent variable.

The first-order ODE (16) is invariant under the first prolongation (extension) of V_8. The canonical coordinates calculated from V_8 are v and u/v^2. In these coordinates the first-order ODE is separable. Also the ODE is linear if u is the dependent variable. The result for u is

$$u = v^2 \int \frac{g(v)\,\mathrm{d}v}{v} + C_1 v^2. \tag{18}$$

Further integration is possible only if equation (18) can be inverted for v in terms of u which produces $y' = h(y)$ from which an implicit function of x is found as an integral. Parametric expressions for x and y in terms of integrals (one is double) over the parameter v are also possible.

The significant result of this section is that a second-order ODE invariant under only one point symmetry has a new type II hidden symmetry that is a contact symmetry. With this type II hidden symmetry the second-order ODE reduces to a first-order ODE which has an additional point symmetry over that predicted by the Lie classical method for point symmetries of the second-order ODE. As a result the second-order ODE can be reduced to quadratures or parametric relations for the two variables x and y can be found. The direct determination of the contact symmetries of a second-order ODE is difficult since a single determining equation, a PDE, is found. However, the realization that contact symmetries may be present and can transform to point symmetries in the reduced first-order ODE is important.

5. Reduction of ODE invariant under one point and two contact symmetries

The third-order ODE, invariant under a three-parameter subgroup with a three-dimensional subalgebra of the three group generators: G_4 (point), G_8 (contact) and G_9 (contact) in equation (10), has the solved form

$$y''' = \frac{y''^3}{y'^4} g\left(\frac{y'^2}{y''} - 2y\right) \tag{19}$$

for g an arbitrary function of its argument. By the Lie classical method this ODE has only one point symmetry, that of G_4. Nevertheless, if this third-order ODE is reduced by the invariants of G_4, the resultant second-order ODE has a point symmetry from the transformation of the contact symmetry of G_8 to a point symmetry of the reduced ODE. We know that the reduced form of G_8 is local as G_4 and G_8 commute. On the other hand G_9 transforms to a non-local group generator since the commutation relations for the three group generators are applicable to either case in equations (2) or (3) with C^i_{jk} or $C^k_{jk} = 0$ respectively. As the reduced form of G_8 is local and the subalgebra of G_4, G_8 and G_9 is of the form in equations (2) or (3), then reduction of the second-order ODE to a first-order

ODE by the invariants of the reduced G_8 gives a local reduced group generator for the twice reduced G_9.

The first reduction of equation (19) is done with the invariants of G_4 which are $u = y$ and $v = y'$. The reduced ODE is

$$v^2\ddot{v} + v\dot{v}^2 = \frac{\dot{v}^3}{v} g\left(\frac{v}{\dot{v}} - 2u\right). \tag{20}$$

The reduced group generators are

$$V_8 = \frac{1}{2}v^2\frac{\partial}{\partial u} \qquad V_9 = v^2\int\frac{\mathrm{d}u}{v}\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v} \tag{21}$$

where as predicted from the Lie algebra V_8 and V_9 are point and linear non-local group generators respectively. The second-order ODE is invariant under the second prolongations of V_8 and V_9.

The second reduction is done by the invariants of V_8 which are

$$z = v \qquad w = \frac{1}{\dot{v}} - \frac{2u}{v}.$$

The first-order ODE is

$$z^3\frac{\mathrm{d}w}{\mathrm{d}z} + z^2 w = -g(zw). \tag{22}$$

The first extension of V_9 is

$$\begin{aligned} V_9^{(1)} &= v^2\int\frac{\mathrm{d}u}{v}\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v} + \left(v\dot{v} - 2v\dot{v}^2\int\frac{\mathrm{d}u}{v}\right)\frac{\partial}{\partial\dot{v}} \\ &= -v^2\int\frac{\mathrm{d}u}{v}\frac{2\partial}{v\partial w} + v^2\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial w} - \left(v\dot{v} - 2v\dot{v}^2\int\frac{\mathrm{d}u}{v}\right)\frac{1}{\dot{v}^2}\frac{\partial}{\partial w}. \end{aligned} \tag{23}$$

The reduced form of $V_9^{(1)}$ is U_9,

$$U_9 = z^2\frac{\partial}{\partial z} - zw\frac{\partial}{\partial w} \tag{24}$$

which is now a local group generator as predicted. The first-order ODE is invariant under the first prolongation of U_9. This example is particularly illuminating since two classes of type II hidden symmetries occur: the transformation of a contact symmetry to a point symmetry and the transformation of a linear non-local symmetry to a point symmetry.

A physically motivated equation which fits into the context of the examples considered here is found in the Langevin equation, which describes the non-relativistic one-dimensional motion of a particle in the presence of radiation, given by [26, 7]

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\ddot{x} = eE + F + m\tau\dddot{x} \tag{25}$$

where $\tau = 2e^2/3mc^3$, m is the mass of the particle, e is its charge, $F = F(x, t)$, a given external force and E is the electric field of the radiation. If the external force is linear in the displacement, i.e. $F = -Kx$, [27] shows that the equation

$$\dddot{x} = \frac{\ddot{x}}{\tau} + \frac{Kx}{m\tau} - \frac{eE(t)}{m\tau} \tag{26}$$

possesses five Lie point symmetries in general. The Lie algebra of these five symmetries is $3A_1 \oplus A_1 \oplus A_1$ [28]. In the case that $2m = -27\tau^2K$ the number of symmetries increases to ten contact symmetries. Consequently the methods of reduction as discussed above can be used for (26). The presence or not of the forcing term does not affect the number of symmetries although it does increase the complexity of their expressions.

6. Conclusions

The usefulness of contact transformations for reducing the order of ODEs has been demonstrated. The contact transformations have frequently been dismissed as curiosities but they are fundamental in evaluating the symmetries of third-order ODEs and reducing the order of third-order ODEs. The evaluation of the contact symmetries of third-order ODEs is now especially easy with the realization that the third-order ODE may be replaced by an equivalent set of lower-order ODEs in the independent variable, the dependent variable and the first derivative for which the point symmetries are the contact symmetries of the third-order ODE. The advent of symbolic computations of point symmetries adds to the ease of determination of the contact symmetries.

The reduction of order of the third-order ODE, $y''' = 0$, by a combination of contact and point symmetries is similar to that for the reduction of ODEs by point symmetries. The higher-dimensional Lie algebra (ten-dimensional) for the contact symmetries than for the point symmetries (seven-dimensional) offers more possibilities for reduction of the ODE. Type I and type II hidden symmetries as were found in the reduction of ODEs by point symmetries occur with contact symmetries. The significant new result is the identification of a new variety of type II hidden symmetry where the contact symmetry transforms to a point symmetry upon reduction of order of the ODE.

Since this new type II hidden symmetry may occur in other third-order and also second-order ODEs, we have analysed two examples of ODEs invariant under a subgroup of the ten-parameter Lie group. The second-order ODE is invariant under one point and one contact symmetry which has an Abelian Lie algebra. It has the new type II hidden symmetry since the contact symmetry transforms to a point symmetry upon reduction of the ODE to first order by the invariants of the point symmetry. The significance of this example is that second-order ODEs may possess contact symmetries which transform to point symmetries upon reduction of order of the ODE but the existence of the contact symmetries may be difficult to detect.

The third-order ODE is invariant under one point symmetry and two contact symmetries. Again the Lie algebra is solvable and the contact symmetries successively transform to point symmetries. The third-order ODE has the new type II hidden symmetry which is a contact symmetry and the second-order ODE has the type II hidden symmetry which is a linear non-local symmetry. The latter is a double type II hidden symmetry since one contact symmetry upon the first reduction becomes a linear non-local symmetry and then transforms to a point symmetry upon the second reduction.

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